

Fractional Calculus Results Based on Extended Gauss Hypergeometric Functions and Integral Transform

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Abstract

Extensions of number of well-known special function such as Beta and Gauss hypergeometric and their properties have been investigated recently by several authors. Our approach is based on the use of Generalized Fractional Calculus (GFC) operators. We aim to investigate the MSM (Marichev-Saigo-Maeda)-fractional calculus operator, Caputo-type MSM-fractional differential operator and pathway fractional integral operator of the extended generalized Gauss hypergeometric function. Furthermore, by employing some integral transform on the resulting formulas, we presented some more image formulas. All the results derived here are of general character and can yield a number of (known and new) results in theory of special functions.

Keywords: Gamma function, extended generalized beta functions, generalized hypergeometric functions, Extended generalized hypergeometric functions, Fractional Integral operators, integral transforms, Pathway fractional integral operator.

Introduction

Throughout this paper $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the sets of positive integers, real numbers, complex numbers and non-positive integers, respectively and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Extensions of a number of well known special functions were investigated by many authors (see, e.g. [1], [2], [4], [5], [6], [8] and see also, very recent work [7]). In particular, Chaudhary et al. [5] gave the following interesting extension of the classical Beta function $B(\alpha, \beta)$:

$$B(\alpha, \beta; \mathcal{P}) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \exp\left(-\frac{\mathcal{P}}{t(1-t)}\right) dt \quad \dots (1.1)$$

$$(\min\{\Re(\alpha), \Re(\beta)\} > 0; \Re(\mathcal{P}) \geq 0)$$

where the Beta function $B(\alpha, \beta; 0) = B(\alpha, \beta)$ is a function of two variables α and β defined by

$$B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt & (\Re(\alpha) > 0, \Re(\beta) > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases} \quad \dots (1.2)$$

and Γ is the familiar Gamma function.

In the sequel, in 2004, by making use of $B_{\mathcal{P}}(x, y)$, Chaudhary et al. [6] extended the Gauss's hypergeometric function as follows:

$$\mathcal{F}_{\mathcal{P}}(a, \ell; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}(\ell + n, c - \ell) z^n}{B(\ell, c - \ell) n!} \quad \dots (1.3)$$

$$(\Re(\mathcal{P}) \geq 0; |z| < 1; \Re(c) > R(\ell) > 0),$$

where $(\lambda)_n$ is the pochhammer symbol or the shifted factorial, which is defined (for $\lambda \in \mathbb{C}$) by (see [11]):

$$(\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \dots (\lambda + n - 1) & (n \in \mathbb{N}) \end{cases}$$

$$= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \quad \dots (1.4)$$

Among several interesting and potentially useful properties for extended hypergeometric function $\mathcal{F}_{\mathcal{P}}(a, \ell; c; z)$ defined by (1.3), the following integral representing was also given by Chaudhary et al. [6]:

$$\mathcal{F}_{\mathcal{P}}(a, \ell; c; z) = \frac{1}{B(\ell, c - \ell)} \int_0^1 t^{\ell-1} (1-t)^{c-\ell-1} (1-zt)^{-a} \exp\left(-\frac{\mathcal{P}}{t(1-t)}\right) dt \quad \dots (1.5)$$

$$(\Re(\mathcal{P}) \geq 0; \mathcal{P} = 0 \text{ and } |\arg(1-z)| < \pi; \Re(c) > R(\ell) > 0)$$

The generalized hypergeometric series ${}_p\mathcal{F}_q$ is defined by (see [16]):

$${}_p\mathcal{F}_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!} = {}_p\mathcal{F}_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \quad \dots (1.6)$$

Here p and q are positive integers or zero (interpreting an empty product as 1) and we assume that the variable z the numerator parameters $\alpha_1, \dots, \alpha_p$ and the denominator parameters $\beta_1, \dots, \beta_q$ take on complex values, provided that no zeros appear in the denominators of (1.6), that is

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; j, \dots, q) \quad \dots (1.7)$$

Obliviously, for the Gauss hypergeometric function ${}_2\mathcal{F}_1$, we have

$$\mathcal{F}_0(a, \ell; c; z) = {}_2\mathcal{F}_1(a, \ell; c; z).$$

In a similar manner, Özergin et al. [4] introduced the following generalization of (1.1) (see, for detail [4],[22]).

$$B_{\mathcal{P}}^{(\rho, \sigma)}(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} {}_1\mathcal{F}_1\left(\rho; \sigma; -\frac{\mathcal{P}}{t(1-t)}\right) dt \quad \dots (1.8)$$

$$(\min\{\Re(\alpha), \Re(\beta), \Re(\rho), \Re(\sigma)\} > 0; \Re(\mathcal{P}) \geq 0)$$

and

$$\mathcal{F}_{\mathcal{P}}^{(\rho, \sigma)}(a, \ell; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(\rho, \sigma)}(\ell + n, c - \ell) z^n}{B(\ell, c - \ell) n!} \quad \dots (1.9)$$

$$(\Re(\mathcal{P}) \geq 0; |z| < 1; \min\{\Re(\rho), \Re(\sigma)\} > 0; \Re(c) > R(\ell) > 0).$$

The following integral representation of the Pfaff-Kummer type was given by Özgerin et al. [4]:

$$\mathcal{F}_{\mathcal{P}}^{(\rho, \sigma)}(a, \ell; c; z) = \frac{1}{B(\ell, c - \ell)} \int_0^1 t^{\ell-1} (1-t)^{c-\ell-1} (1-zt)^{-\alpha} {}_1F_1\left(\rho; \sigma; -\frac{\mathcal{P}}{t(1-t)}\right) dt$$

... (1.10)

$$(\Re(\mathcal{P}) \geq 0; \mathcal{P} = 0 \text{ and } |\arg(1-z)| < \pi; \Re(c) > R(\ell) > 0; \min\{\Re(\rho), \Re(\sigma)\} > 0).$$

Very recently, Srivastava et al. [2] introduced a further natural generalization of (1.8) and (1.9), respectively, in terms of the function $\Xi(\{k_\ell\}_{\ell \in \mathbb{N}_0}; z)$ defined as follows (see [2]):

$$B_{\mathcal{P}}^{(\{k_\ell\}_{\ell \in \mathbb{N}_0})}(x, y; \mathcal{P}) = \int_0^1 t^{x-1} (1-t)^{y-1} \Xi\left(\{k_\ell\}_{\ell \in \mathbb{N}_0}; -\frac{\mathcal{P}}{t(1-t)}\right) dt$$

... (1.11)

$$(\Re(\mathcal{P}) \geq 0; \min\{\Re(x), \Re(y)\} > 0).$$

and

$$\mathcal{F}_{\mathcal{P}}^{(\{k_\ell\}_{\ell \in \mathbb{N}_0})}(a, \ell; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(\{k_\ell\}_{\ell \in \mathbb{N}_0})}(\ell + n, c - \ell; \mathcal{P}) z^n}{B(\ell, c - \ell) n!}$$

... (1.12)

$$(|z| < 1; \Re(c) > R(\ell) > 0; \Re(\mathcal{P}) \geq 0)$$

where $\Xi(\{k_\ell\}_{\ell \in \mathbb{N}_0}; z)$ is given by the following definition (see [2]):

Definition: Let a function $\Xi(\{k_\ell\}_{\ell \in \mathbb{N}_0}; z)$ be analytic within disk $|z| < \Re$ ($0 < \Re < \infty$) and let its Tylor-Maclaurin coefficients be explicitly denoted by the sequence $\{k_\ell\}_{\ell \in \mathbb{N}_0}$. Suppose also that the function $\Xi(\{k_\ell\}_{\ell \in \mathbb{N}_0}; z)$ can be continued analytically in the right half plane $\Re(z) > 0$ with the asymptotic properties given as follows:

$$\Xi(\{k_\ell\}_{\ell \in \mathbb{N}_0}; z) = \begin{cases} \sum_{\ell=0}^{\infty} k_\ell \frac{z^\ell}{\ell!} & (|z| < \Re; 0 < \Re < \infty; k_0 = 1) \\ \mathcal{M}_0 z^\omega \exp(z) \left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right] & (\Re(z) \rightarrow \infty; \mathcal{M}_0 > 0; \omega \in \mathbb{C}) \end{cases}$$

... (1.13)

for the suitable constants $\mathcal{M}_0$ and ω depending on essentially upon the sequence $\{k_\ell\}_{\ell \in \mathbb{N}_0}$. Here we assume that the series in the first part of the definition (1.13) converges absolutely when $|z| < \Re$ for some $\Re$ ($0 < \Re < \infty$) and represents the function $\Xi(\{k_\ell\}_{\ell \in \mathbb{N}_0}; z)$ which is assumed to be analytic within the disk $|z| < \Re$ for some

$\Re(0 < \Re < \infty)$ and which can be appropriately continued analytically elsewhere in complex z -plane with the order estimate provided in the second part of the definition (1.13).

The outlined above-mentioned detailed and systematic investigation was indeed motivated largely by the demonstrated potential for applications of the more extended generalized Gauss hypergeometric function and their special cases in many diverse areas of mathematical, physical, engineering, and statistical science (see [2]).

Let $\lambda, \lambda', \zeta, \zeta', \gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$ and $x \in \mathbb{R}^+$. Then the generalized fractional integral operators involving the Appell functions F_3 are as follows:

$$\left(I_{0,+}^{\lambda, \lambda', \zeta, \zeta', \gamma} f\right)(x) = \frac{x^{-\lambda}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\lambda'} F_3\left(\lambda, \lambda', \zeta, \zeta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t) dt \quad \dots (1.14)$$

and

$$\left(I_{-}^{\lambda, \lambda', \zeta, \zeta', \gamma} f\right)(x) = \frac{x^{-\lambda'}}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} t^{-\lambda} F_3\left(\lambda, \lambda', \zeta, \zeta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t) dt \quad \dots (1.15)$$

The generalized fractional integral operators of types (1.14) and (1.15) have been introduced by Marichev [9] and later extended and studied by Saigo and Maeda [10]. These operators are known as the Marichev-Saigo-Maeda operators (MSM-Operators). Recently, Mondal and Nisar [17], Nisar et al. [19] have been investigated the Marichev-Saigo-Maeda operators generalized Bessel function and k -Mittag Leffler function (see also [18]).

The corresponding fractional differential operators have their respective forms:

$$\left(D_{0,+}^{\lambda, \lambda', \zeta, \zeta', \gamma} f\right)(x) = \left(\frac{d}{dx}\right)^{[\Re(\gamma)]+1} \left(I_{0,+}^{-\lambda, -\lambda', -\zeta', -\zeta'+[\Re(\gamma)]+1, -\gamma+[\Re(\gamma)]+1} f\right)(x) \quad \dots (1.16)$$

and

$$\left(D_{-}^{\lambda, \lambda', \zeta, \zeta', \gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^{[\Re(\gamma)]+1} \left(I_{-}^{-\lambda, -\lambda', -\zeta', -\zeta'+[\Re(\gamma)]+1, -\gamma+[\Re(\gamma)]+1} f\right)(x) \quad \dots (1.17)$$

The fractional integral operators have many interesting applications in various fields. For some results on fractional calculus, we refer to ([20, 21]).

The following four results will be required (for the first and second, see [3]; for the third and fourth, see [12]).

Lemma 1. Let $\lambda, \lambda', \zeta, \zeta', \gamma, \rho \in \mathbb{C}$ be such that $\Re(\gamma) > 0$ and $\Re(\rho) > \max\{0, \Re(\lambda + \lambda' + \zeta - \gamma), \Re(\lambda' - \zeta')\}$.

Then

$$\left(I_{0,+}^{\lambda, \lambda', \zeta, \zeta', \gamma} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\lambda-\lambda'-\zeta)\Gamma(\rho+\zeta'-\lambda')}{\Gamma(\rho+\zeta')\Gamma(\rho+\gamma-\lambda-\lambda')\Gamma(\rho+\gamma-\lambda'-\zeta)} x^{\rho-\lambda-\lambda'+\gamma-1} \quad \dots (1.18)$$

Lemma 2. Let $\lambda, \lambda', \zeta, \zeta', \gamma, \rho \in \mathbb{C}$ be such that $\Re(\gamma) > 0$ and $\Re(\rho) > \max\{\Re(\zeta), \Re(-\lambda - \lambda' + \gamma), \Re(-\lambda - \zeta' + \gamma)\}$. Then

$$(I_{-}^{\lambda, \lambda', \zeta, \zeta', \gamma} t^{-\rho})(x) = \frac{\Gamma(-\zeta+\rho)\Gamma(\lambda+\lambda'-\gamma+\rho)\Gamma(\lambda+\zeta'-\gamma+\rho)}{\Gamma(\rho)\Gamma(\lambda-\zeta+\rho)\Gamma(\lambda+\lambda'+\zeta'-\gamma+\rho)} x^{-\lambda-\lambda'+\gamma-\rho} \quad \dots (1.19)$$

Lemma 3. Let $\lambda, \delta, \gamma, \rho \in \mathbb{C}$ with $\Re(\lambda) > 0$ and $\Re(\rho) > \max\{0, -\Re(\delta - \gamma)\}$. Then

$$(I_{0+}^{\lambda, \delta, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\delta)}{\Gamma(\rho-\delta)\Gamma(\rho+\lambda+\gamma)} x^{\rho-\delta-1} \quad \dots (1.20)$$

In Particular

$$(I_{0+}^{\lambda, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho+\gamma)}{\Gamma(\rho+\lambda+\gamma)} x^{\rho-1} \quad \dots (1.21)$$

$$(\Re(\lambda) > 0, \Re(\rho) > \max\{0, -\Re(\gamma)\})$$

and

$$(I_0^{\lambda} t^{\rho-1})(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\lambda)} x^{\rho-1} \quad (\Re(\lambda) > 0, \Re(\rho) > 0) \quad \dots (1.22)$$

Lemma 4. Let $\lambda, \delta, \gamma, \rho \in \mathbb{C}$ with $\Re(\lambda) > 0$ and $\Re(\rho) < 1 + \min\{\Re(\delta), \Re(\gamma)\}$. Then

$$(I_{-}^{\lambda, \delta, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\delta-\rho+1)\Gamma(\gamma-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\delta+\gamma-\rho+1)} x^{\rho-\delta-1} \quad \dots (1.23)$$

In Particular

$$(I_{-}^{\lambda, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\gamma-\rho+1)}{\Gamma(\lambda+\gamma-\rho+1)} x^{\rho-1} \quad \dots (1.24)$$

$$(\Re(\lambda) > 0, \Re(\rho) < 1 + \Re(\gamma))$$

and

$$(I_{-}^{\lambda} t^{\rho-1})(x) = \frac{\Gamma(1-\rho)}{\Gamma(\lambda-\rho+1)} x^{\rho-1} \quad \dots (1.25)$$

$$(\Re(\lambda) > 0, \Re(\rho) < 1)$$

MSM Fractional Integral Representation of the Extended Generalized Hypergeometric Function

Here, in this section, we shall establish some fractional integral formulas for the extended generalized Gauss hypergeometric type functions $\mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{E}})}(\cdot)$.

Theorem 1. Let $x > 0$, $\lambda, \lambda', \zeta, \zeta', \gamma, \rho, e \in \mathbb{C}$ with $\Re(c) > \Re(b) > 0, \Re(\mathcal{P}) \geq 0$ and $\Re(\rho) > \max\{0, \Re(\lambda + \lambda' + \zeta - \gamma), \Re(\lambda' - \zeta')\}$. Then the following integral formula holds true:

$$(I_{0,+}^{\lambda, \lambda', \zeta, \zeta', \gamma} [t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{E}})}(a, b; c; et)])(x) = t^{\rho-\lambda-\lambda'+\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\lambda-\lambda'-\zeta)\Gamma(\rho+\zeta'-\lambda')}{\Gamma(\rho+\zeta')\Gamma(\rho+\gamma-\lambda-\lambda')\Gamma(\rho+\gamma-\lambda'-\zeta)}$$

$$\times {}_3\mathcal{F}_{\mathcal{P}+3}^{(k_{\mathcal{I}})} \left[\begin{matrix} a, \mathfrak{b}, \rho, (\rho + \gamma - \lambda - \lambda' - \zeta), (\rho + \zeta' - \lambda'); \\ c, (\rho + \zeta'), (\rho + \gamma - \lambda - \lambda'), (\rho + \gamma - \lambda' - \zeta); \end{matrix} ex \right] \quad (|x| < 1) \quad \dots (2.1)$$

Proof: For convenience and simplicity, we denote the left hand side of (2.1) by $\mathfrak{T}$. Then applying (1.12), we have

$$\mathfrak{T} = \left(I_{0,+}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{I}})}(\mathfrak{b} + n, c - \mathfrak{b}; \mathcal{P})}{B(\mathfrak{b}, c - \mathfrak{b})} \frac{e^n t^n}{n!} \right] \right) (x)$$

Interchanging the summation and integration, which is valid under the condition of Theorem 1, we find that

$$\mathfrak{T} = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{I}})}(\mathfrak{b} + n, c - \mathfrak{b}; \mathcal{P})}{B(\mathfrak{b}, c - \mathfrak{b})} \frac{e^n}{n!} \left(I_{0,+}^{\lambda, \lambda', \zeta, \zeta', \gamma} [t^{\rho+n-1}] \right) (x)$$

Applying Lemma 1, we get

$$\begin{aligned} \mathfrak{T} &= \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{I}})}(\mathfrak{b} + n, c - \mathfrak{b}; \mathcal{P})}{B(\mathfrak{b}, c - \mathfrak{b})} \frac{e^n}{n!} \frac{\Gamma(\rho + n) \Gamma(\rho + n + \gamma - \lambda - \lambda' - \zeta)}{\Gamma(\rho + n + \zeta') \Gamma(\rho + n + \gamma - \lambda - \lambda')} \\ &\quad \times \frac{\Gamma(\rho + n + \zeta' - \lambda')}{\Gamma(\rho + n + \gamma - \lambda' - \zeta)} t^{\rho+n-\lambda-\lambda'+\gamma-1} \\ &= t^{\rho-\lambda-\lambda'+\gamma-1} \frac{\Gamma(\rho) \Gamma(\rho + \gamma - \lambda - \lambda' - \zeta) \Gamma(\rho + \zeta' - \lambda')}{\Gamma(\rho + \zeta') \Gamma(\rho + \gamma - \lambda - \lambda') \Gamma(\rho + \gamma - \lambda' - \zeta)} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{I}})}(\mathfrak{b} + n, c - \mathfrak{b}; \mathcal{P})}{B(\mathfrak{b}, c - \mathfrak{b})} \\ &\quad \times \frac{(\rho)_n (\rho + \gamma - \lambda - \lambda' - \zeta)_n (\rho + \zeta' - \lambda')_n}{(\rho + \zeta')_n (\rho + \gamma - \lambda - \lambda')_n (\rho + \gamma - \lambda' - \zeta)_n} \frac{(ex)^n}{n!}. \end{aligned}$$

Which, in view of (1.12), leads to the right-hand side of (2.1). This completes the proof.

If we reduce MSM fractional integral in the Saigo fractional integral formula, then we arrive at the following result recently obtained by Choi et al. [7].

Corollary 1. Let $x > 0, \Re(\mathcal{P}) \geq 0, \Re(c) > R(\mathfrak{b}) > 0$ with the parameters $\lambda, \zeta, \gamma, \rho, e \in \mathbb{C}$ satisfying the inequality $\Re(\rho) > \max\{0, \Re(\zeta - \gamma)\}$. Then the following fractional integral formula holds true:

$$\begin{aligned} &\left(I_{0,+}^{\lambda, \zeta, \gamma} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{I}})}(a, \mathfrak{b}; c; et) \right] \right) (x) = t^{\rho-\zeta-1} \frac{\Gamma(\rho) \Gamma(\rho - \zeta + \gamma)}{\Gamma(\rho - \zeta) \Gamma(\rho + \lambda + \gamma)} \\ &\times {}_2\mathcal{F}_{\mathcal{P}+2}^{(k_{\mathcal{I}})} \left[\begin{matrix} a, \mathfrak{b}, \rho, (\rho - \zeta + \gamma); \\ c, (\rho - \zeta), (\rho + \lambda + \gamma); \end{matrix} ex \right] \quad (|x| < 1) \quad \dots (2.2) \end{aligned}$$

Theorem 2. Let $\lambda, \lambda', \zeta, \zeta', \gamma, \rho, e \in \mathbb{C}$ with $\rho > 0, \Re(c) > R(\mathfrak{b}) > 0, \Re(\mathcal{P}) \geq 0$ be such that $\Re(\rho) > \max\{\Re(\zeta), \Re(-\lambda - \lambda' + \gamma), \Re(-\lambda - \zeta' + \gamma)\}$. Then the following integral formula holds true:

$$\left(I_{0,+}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{I}})} \left(a, \mathfrak{b}; c; \frac{e}{t} \right) \right] \right) (x) = t^{\rho-\lambda-\lambda'+\gamma-1} \frac{\Gamma(1-\rho-\zeta) \Gamma(1-\rho+\lambda+\lambda'-\zeta) \Gamma(1-\rho+\lambda+\zeta'-\gamma)}{\Gamma(1-\rho) \Gamma(1-\rho+\lambda-\zeta) \Gamma(1-\rho+\lambda+\lambda'+\zeta'-\gamma)}$$

$$\times {}_3\mathcal{F}_{\mathcal{P}+3}^{(k_{\mathcal{P}})} \left[\begin{matrix} a, \mathfrak{b}, (1-\rho-\zeta), (1-\rho+\lambda+\lambda'-\zeta), (1-\rho+\lambda+\zeta'-\gamma); \\ c, (1-\rho), (1-\rho+\lambda-\zeta), (1-\rho+\lambda+\lambda'+\zeta'-\gamma); \end{matrix} \frac{e}{x} \right] \quad \dots (2.3)$$

Proof. We can establish (2.3) by a similar argument as in the proof of (2.1), using Lemma (2) instead of Lemma (1). Therefore, we omit its details.

Interestingly, if we reduce MSM fractional integral operator in Saigo fractional integral operator, then, we arrive at the following result obtained by Choi et al. [7].

Corollary 2. Let $x > 0, \Re(\mathcal{P}) \geq 0, \Re(c) > R(\mathfrak{b}) > 0$ with the parameters $\lambda, \zeta, \gamma, \rho, e \in \mathbb{C}$ satisfying the inequality $\Re(\rho) > \max\{\Re(-\zeta), \Re(-\gamma)\}$. Then we have

$$\left(I_{-}^{\lambda, \zeta, \gamma} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})} \left(a, \mathfrak{b}; c; \frac{e}{t} \right) \right] \right) (x) = t^{\rho-\zeta-1} \frac{\Gamma(1-\rho+\zeta)\Gamma(1-\rho+\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho+\lambda+\zeta+\gamma)} \\ \times {}_2\mathcal{F}_{\mathcal{P}+2}^{(k_{\mathcal{P}})} \left[\begin{matrix} a, \mathfrak{b}, \rho, (1-\rho+\zeta), (1-\rho+\gamma); \\ c, \Gamma(1-\rho), \Gamma(1-\rho+\lambda+\zeta+\gamma); \end{matrix} \frac{e}{x} \right] \quad \dots (2.4)$$

MSM Fractional Differential Representation of the Extended Generalized Hypergeometric Function

In this section, we shall establish the Marichev-Saigo-Maeda fractional differentiation of the extended generalized Gauss hypergeometric type functions $\mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})}(\cdot)$. For our purpose, the following Lemmas will be required (see [23]).

Lemma 5. Let $\lambda, \lambda', \zeta, \zeta', \gamma, \rho \in \mathbb{C}$ be such that $\Re(\rho) > \max\{0, \Re(-\lambda+\zeta), \Re(-\lambda-\lambda'-\zeta'+\gamma)\}$. Then

$$\left(D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} t^{\rho-1} \right) (x) \\ = \frac{\Gamma(\rho)\Gamma(-\zeta+\lambda+\rho)\Gamma(\lambda+\lambda'+\zeta'-\gamma+\rho)}{\Gamma(-\zeta+\rho)\Gamma(\lambda+\lambda'-\gamma+\rho)\Gamma(\lambda+\zeta'-\gamma+\rho)} x^{\lambda+\lambda'-\gamma+\rho-1} \quad \dots (3.1)$$

Lemma 6. Let $\lambda, \lambda', \zeta, \zeta', \gamma, \rho \in \mathbb{C}$ be such that $\Re(\rho) > \max\{\Re(-\zeta'), \Re(\lambda'+\zeta-\gamma), \Re(\lambda+\lambda'-\gamma)+[\Re(\gamma)]+1\}$. Then

$$\left(D_{-}^{\lambda, \lambda', \zeta, \zeta', \gamma} t^{-\rho} \right) (x) \\ = \frac{\Gamma(\zeta'+\rho)\Gamma(-\lambda-\lambda'+\gamma+\rho)\Gamma(-\lambda'-\zeta+\gamma+\rho)}{\Gamma(\rho)\Gamma(-\lambda'+\zeta'+\rho)\Gamma(-\lambda-\lambda'-\zeta+\gamma+\rho)} x^{\lambda+\lambda'-\gamma-\rho} \quad \dots (3.2)$$

Theorem 3. Let $x > 0, \Re(\mathcal{P}) \geq 0, \Re(c) > R(\mathfrak{b}) > 0$ and the parameters $\lambda, \lambda', \zeta, \zeta', \gamma, \rho, e \in \mathbb{C}$ be such that $\Re(\rho) > \max\{0, \Re(-\lambda+\zeta), \Re(-\lambda-\lambda'-\zeta'+\gamma)\}$. Then we have

$$\left(D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})} (a, \mathfrak{b}; c; et) \right] \right) (x) = x^{\rho+\lambda+\lambda'-\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho-\zeta+\lambda)\Gamma(\rho+\lambda+\lambda'+\zeta'-\gamma)}{\Gamma(\rho-\zeta)\Gamma(\rho+\lambda+\lambda'-\gamma)\Gamma(\rho+\lambda+\zeta'-\gamma)} \\ \times {}_3\mathcal{F}_{\mathcal{P}+3}^{(k_{\mathcal{P}})} \left[\begin{matrix} a, \mathfrak{b}, \rho, (\rho-\zeta+\lambda), (\rho+\lambda+\lambda'+\zeta'-\gamma); \\ c, \Gamma(\rho-\zeta), \Gamma(\rho+\lambda+\lambda'-\gamma), (\rho+\lambda+\zeta'-\gamma); \end{matrix} ex \right] \quad \dots (3.3)$$

Proof. Let $\Lambda(x)$ be the left-hand side of (3.3) and using (1.12), we have

$$\Lambda(x) = \left(D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P}) e^n t^n}{B(\ell, c - \ell)} \frac{e^n t^n}{n!} \right] \right) (x)$$

Interchanging the summation and differentiation, which is valid under the condition of Theorem 3, we find that

$$\Lambda(x) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P}) e^n}{B(\ell, c - \ell)} \frac{e^n}{n!} \left(D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} [t^{\rho+n-1}] \right) (x)$$

Next applying Lemma 5, we obtain

$$\begin{aligned} \Lambda(x) &= \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P}) e^n}{B(\ell, c - \ell)} \frac{1}{n!} \frac{\Gamma(\rho + n) \Gamma(\rho + n - \zeta + \lambda)}{\Gamma(\rho + n - \zeta) \Gamma(\lambda + \lambda' - \gamma + \rho + n)} \\ &\quad \times \frac{\Gamma(\rho + n + \lambda + \lambda' + \zeta' - \gamma)}{\Gamma(\rho + n + \lambda + \zeta' - \gamma)} t^{\rho+n+\lambda+\lambda'-\gamma-1} \\ &= t^{\rho+\lambda+\lambda'-\gamma-1} \frac{\Gamma(\rho) \Gamma(\rho - \zeta + \lambda) \Gamma(\rho + \lambda + \lambda' + \zeta' - \gamma)}{\Gamma(\rho - \zeta) \Gamma(\rho + \lambda + \lambda' - \gamma) \Gamma(\rho + \lambda + \zeta' - \gamma)} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P})}{B(\ell, c - \ell)} \\ &\quad \times \frac{(\rho)_n (\rho - \zeta + \lambda)_n (\rho + \lambda + \lambda' + \zeta' - \gamma)_n}{(\rho - \zeta)_n (\rho + \lambda + \lambda' - \gamma)_n (\rho + \lambda + \zeta' - \gamma)_n} \frac{(ex)^n}{n!}, \end{aligned}$$

which, in view of (1.12), we readily obtain the right-hand side of (3.3).

Theorem 4. Let $x > 0$, $\Re(\mathcal{P}) \geq 0$, $\Re(c) > R(\ell) > 0$ and the parameters $\lambda, \lambda', \zeta, \zeta', \gamma, \rho \in \mathbb{C}$ be such that $\Re(\rho) > \max\{\Re(-\zeta'), \Re(\lambda' + \zeta' - \gamma), \Re(\lambda + \lambda' - \gamma) + [\Re(\gamma)] + 1\}$. Then we have

$$\begin{aligned} &\left(D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\ell})} \left(a, \ell; c; \frac{e}{t} \right) \right] \right) (x) \\ &= x^{\rho+\lambda+\lambda'-\gamma-1} \frac{\Gamma(1 - \rho + \zeta') \Gamma(1 - \rho - \lambda - \lambda' + \gamma) \Gamma(1 - \rho - \lambda' - \zeta + \gamma)}{\Gamma(1 - \rho) \Gamma(1 - \rho - \lambda' + \zeta) \Gamma(1 - \rho - \lambda - \lambda' - \zeta + \gamma)} \\ &\quad \times {}_3\mathcal{F}_{\mathcal{P}+3}^{(k_{\ell})} \left[\begin{matrix} a, \ell, (1 - \rho + \zeta'), (1 - \rho - \lambda - \lambda' + \gamma), (1 - \rho - \lambda' - \zeta + \gamma); \\ c, \Gamma(1 - \rho), \Gamma(1 - \rho - \lambda' + \zeta), (1 - \rho - \lambda - \lambda' - \zeta + \gamma); \end{matrix} ex \right] \quad \dots (3.4) \end{aligned}$$

Proof. The proof of the fractional integration formula (3.4) would run parallel to the proof of (3.3) by using Lemma (6). So, its proof details are omitted.

Caputo-type MSM Fractional Differentiation Representation of the Extended Generalized Hypergeometric Function

Rao et al. [15] introduced the Caputo-type fractional derivatives that have the Gauss hypergeometric function in the kernel. The Left-and right-hand sided Caputo fractional differential operators associated with the Gauss hypergeometric function are defined, respectively, by

$$\left({}^c D_{0+}^{\lambda, \zeta, \gamma} f\right)(x) = \left(I_{0+}^{-\lambda + [\Re(\lambda)] + 1, -\zeta - [\Re(\lambda)] - 1, \lambda + \gamma - [\Re(\lambda)] - 1} f^{([\Re(\lambda)] + 1)}\right)(x)$$

and

$$\left({}^c D_{-}^{\lambda, \zeta, \gamma} f\right)(x) = (-1)^{[\Re(\lambda)] + 1} \left(I_{-}^{-\lambda + [\Re(\lambda)] + 1, -\zeta - [\Re(\lambda)] - 1, \lambda + \gamma - [\Re(\lambda)] - 1} f^{([\Re(\lambda)] + 1)}\right)(x),$$

where $\lambda, \zeta, \gamma \in \mathbb{C}$ with $\Re(\lambda) > 0$ and $x \in \mathbb{R}^+$.

The Left-and right-hand sided Caputo-type MSM fractional differential operators associated with Appell F_3 are defined, respectively, by

$$\left({}^c D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} f\right)(x) = \left(I_{0+}^{-\lambda', -\lambda, -\zeta' + [\Re(\lambda)] + 1, -\zeta', -\gamma + [\Re(\lambda)] + 1} f^{([\Re(\lambda)] + 1)}\right)(x)$$

and

$$\left({}^c D_{-}^{\lambda, \lambda', \zeta, \zeta', \gamma} f\right)(x) = (-1)^{[\Re(\lambda)] + 1} \left(I_{-}^{-\lambda', -\lambda, -\zeta' + [\Re(\lambda)] + 1, -\zeta', -\gamma + [\Re(\lambda)] + 1} f^{([\Re(\lambda)] + 1)}\right)(x).$$

where $\lambda, \lambda', \zeta, \zeta', \gamma \in \mathbb{C}$ with $\Re(\lambda) > 0$ and $x \in \mathbb{R}^+$.

In this section we derive the left-and right-sided Caputo fractional differential formulas of the extended hypergeometric function (1.12). The following Lemmas given in (see [23]) are needed in sequel.

Lemma 7. Let $\lambda, \lambda', \zeta, \zeta', \gamma, \rho \in \mathbb{C}$ and $m = [\Re(\gamma)] + 1$ with $\Re(\rho) - m > \max\{0, \Re(-\lambda + \zeta), (-\lambda - \lambda' - \zeta' + \gamma)\}$. Then

$$\begin{aligned} & \left({}^c D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} t^{\rho-1}\right)(x) \\ &= \frac{\Gamma(\rho)\Gamma(\lambda - \zeta + \rho - m)\Gamma(\lambda + \lambda' + \zeta' - \gamma + \rho - m)}{\Gamma(-\zeta + \rho - m)\Gamma(\lambda + \lambda' - \gamma + \rho)\Gamma(\lambda + \zeta' - \gamma + \rho - m)} x^{\lambda + \lambda' - \gamma + \rho + 1} \dots (4.1) \end{aligned}$$

Lemma 8. Let $\lambda, \lambda', \zeta, \zeta', \gamma, \rho \in \mathbb{C}$ and $m = [\Re(\gamma)] + 1$ with $\Re(\rho) + m > \max\{\Re(-\zeta'), \Re(\lambda' + \zeta - \gamma), \Re(\lambda + \lambda' - \gamma) + m\}$. Then

$$\begin{aligned} & \left({}^c D_{-}^{\lambda, \lambda', \zeta, \zeta', \gamma} t^{-\rho}\right)(x) \\ &= \frac{\Gamma(\zeta' + \rho + m)\Gamma(-\lambda - \lambda' + \gamma + \rho)\Gamma(-\lambda' + \zeta' + \gamma + \rho + m)}{\Gamma(\rho)\Gamma(-\lambda' + \zeta' + \rho + m)\Gamma(-\lambda - \lambda' - \zeta + \gamma + \rho + m)} x^{\lambda + \lambda' - \gamma - \rho} \dots (4.2) \end{aligned}$$

Theorem 5. Let $x > 0, \Re(\mathcal{P}) \geq 0, \Re(c) > R(\mathfrak{b}) > 0, m = [\Re(\gamma)] + 1$ and the parameters $\lambda, \lambda', \zeta, \zeta', \gamma, \rho, e \in \mathbb{C}$ be such that $\Re(\rho) - m > \max\{0, \Re(-\lambda + \zeta), (-\lambda - \lambda' - \zeta' + \gamma)\}$. Then there holds the following formula:

$$\begin{aligned} & \left({}^c D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})}(a, \mathfrak{b}; c; et) \right] \right) (x) \\ &= x^{\lambda+\lambda'-\gamma+\rho+1} \frac{\Gamma(\rho)\Gamma(\rho+\lambda-\zeta-m)\Gamma(\rho+\lambda+\lambda'+\zeta'-\gamma-m)}{\Gamma(\rho-\zeta-m)\Gamma(\rho+\lambda+\lambda'-\gamma)\Gamma(\rho+\lambda+\zeta'-\gamma-m)} \\ & \times {}_3\mathcal{F}_{\mathcal{P}+3}^{(k_{\mathcal{P}})} \left[\begin{matrix} a, \mathfrak{b}, (\rho), (\rho+\lambda-\zeta-m), (\rho+\lambda+\lambda'+\zeta'-\gamma-m); \\ c, \Gamma(\rho-\zeta-m), \Gamma(\rho+\lambda+\lambda'-\gamma), (\rho+\lambda+\zeta'-\gamma-m); \end{matrix} e x \right] \quad \dots (4.3) \end{aligned}$$

Proof. For convenience, we denote the left-hand side of (4.3) by $\Lambda(x)$. Applying (1.12), we get

$$\Lambda(x) = \left({}^c D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{P}})}(\mathfrak{b}+n, c-\mathfrak{b}; \mathcal{P}) e^n t^n}{B(\mathfrak{b}, c-\mathfrak{b})} \frac{e^n t^n}{n!} \right] \right) (x)$$

Interchanging the order of summation and differentiation, which is guaranteed under the given condition of in this theorem, we find that

$$\Lambda(x) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{P}})}(\mathfrak{b}+n, c-\mathfrak{b}; \mathcal{P}) e^n}{B(\mathfrak{b}, c-\mathfrak{b})} \frac{e^n}{n!} \left({}^c D_{0+}^{\lambda, \lambda', \zeta, \zeta', \gamma} [t^{\rho+n-1}] \right) (x)$$

Next using Lemma 7, we obtain

$$\begin{aligned} \Lambda(x) &= \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{P}})}(\mathfrak{b}+n, c-\mathfrak{b}; \mathcal{P}) e^n}{B(\mathfrak{b}, c-\mathfrak{b})} \frac{e^n}{n!} \frac{\Gamma(\rho+n)\Gamma(\rho+\lambda-\zeta+n-m)}{\Gamma(\rho-\zeta+n-m)\Gamma(\rho+\lambda+\lambda'-\gamma+n)} \\ & \times \frac{\Gamma(\rho+\lambda+\lambda'+\zeta'-\gamma+n-m)}{\Gamma(\rho+\lambda+\zeta'-\gamma+n-m)} t^{\rho+n+\lambda+\lambda'-\gamma-1} \end{aligned}$$

Interpreting the right-hand side of the above equation, in the view of the definition (1.12), we arrive at the required result (4.3).

Theorem 6. Let $x > 0, \Re(\mathcal{P}) \geq 0, \Re(c) > R(\mathfrak{b}) > 0, m = [\Re(\gamma)] + 1$ and the parameters $\lambda, \lambda', \zeta, \zeta', \gamma, \rho, e \in \mathbb{C}$ be such that $\Re(\rho) + m > \max\{\Re(-\zeta'), \Re(\lambda + \lambda' - \gamma) + m\}$. Then there holds the following formula:

$$\begin{aligned} & \left({}^c D_{-}^{\lambda, \lambda', \zeta, \zeta', \gamma} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})} \left(a, \mathfrak{b}; c; \frac{e}{t} \right) \right] \right) (x) \\ &= x^{\lambda+\lambda'-\gamma+\rho-1} \frac{\Gamma(1+\zeta'-\rho+m)\Gamma(1+\gamma-\lambda-\lambda'-\rho)\Gamma(1+\gamma-\lambda'-\zeta-\rho+m)}{\Gamma(1-\rho)\Gamma(1-\lambda'+\zeta'-\rho+m)\Gamma(1-\lambda-\lambda'-\zeta+\gamma-\rho+m)} \\ & \times {}_3\mathcal{F}_{\mathcal{P}+3}^{(k_{\mathcal{P}})} \left[\begin{matrix} a, \mathfrak{b}, (1+\zeta'-\rho+m), (1+\gamma-\lambda-\lambda'-\rho), (1+\gamma-\lambda'-\zeta-\rho+m); \\ c, \Gamma(1-\rho), \Gamma(1-\lambda'+\zeta'-\rho+m), (1-\lambda-\lambda'-\zeta+\gamma-\rho+m); \end{matrix} \frac{e}{x} \right] \end{aligned}$$

... (4.4)

Proof. The proof of the result (4.4) follows the same as that employed in the proof of theorem 5 and consequently, we omit the details.

5. Integral Transform of the Extended Generalized Hypergeometric Functions

In this section, the image of $\mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(\cdot)$ under Beta, Laplace and Whittaker functions have been obtained. For our purpose, we begin by recalling some integral transforms.

The Euler (Beta) transform of $f(z)$ is defined as [25]:

$$B\{f(z); \alpha, \beta\} = \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} dz \quad (\Re(\alpha) > 0, \Re(\beta) > 0) \quad \dots (5.1)$$

Laplace transform of $f(z)$ is defined as [25]:

$$\mathcal{L}\{f(z); s\} = \int_0^{\infty} e^{-sz} f(z) dz \quad (\Re(s) > 0) \quad \dots (5.2)$$

The Whittaker transform is defined as [26]:

$$\int_0^{\infty} e^{-t/2} t^{\vartheta-1} \mathcal{W}_{\xi, \mu}(t) dt = \frac{\Gamma\left(\frac{1}{2} + \mu + \vartheta\right) \Gamma\left(\frac{1}{2} - \mu + \vartheta\right)}{\Gamma(1 - \xi + \vartheta)} \quad \dots (5.3)$$

where $\Re(\mu \pm \vartheta) > -1/2$ and $\mathcal{W}_{\xi, \mu}(t)$ is the Whittaker confluent hypergeometric function.

Theorem 7. (Beta function)

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(l+m, \ell; c; yz) : l, m\right\} = B(l, m) \cdot \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(l, \ell; c; y) \quad \dots (5.4)$$

where $\Re(c) < \Re(\ell) > 0, \Re(l) > 0, \Re(m) > 0$ and $\Re(\mathcal{P}) \geq 0$.

Proof. In order to prove (5.4), by using the definition of Beta transform (5.1), the LHS of (5.4) becomes

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(l+m, \ell; c; yz) : l, m\right\} = \int_0^1 z^{l-1} (1-z)^{m-1} \left\{\mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(l+m, \ell; c; yz)\right\} dz$$

Now using the definition (1.12), we get

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(l+m, \ell; c; yz) : l, m\right\} = \int_0^1 z^{l-1} (1-z)^{m-1} \sum_{n=0}^{\infty} (l+m)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell+n, c-\ell; \mathcal{P}) (yz)^n}{B(\ell, c-\ell) n!} dz$$

Interchanging the order of integration and summation and using beta integral, we get

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(l+m, \ell; c; yz) : l, m\right\}$$

$$= \sum_{n=0}^{\infty} (l+m)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell+n, c-\ell; \mathcal{P}) y^n}{B(\ell, c-\ell)} \frac{1}{n!} \left\{ \int_0^1 z^{l+n-1} (1-z)^{m-1} dz \right\}$$

$$= \sum_{n=0}^{\infty} (l+m)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell+n, c-\ell; \mathcal{P}) y^n}{B(\ell, c-\ell)} \frac{1}{n!} \frac{\Gamma(l+n)\Gamma(m)}{\Gamma(l+n+m)}$$

Then applying (1.4) and interpreting the right-hand side of the above equation, in the view of the definition (1.12), we arrive at the required result (5.4).

Theorem 8. (Laplace Transform)

$$\mathcal{L} \left\{ z^{l-1} \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; yz); s \right\} = \frac{\Gamma(l)}{s^l} {}_1\mathcal{F}_{\mathcal{P}}^{(k_{\ell})} \left(a, l, \ell; c; \frac{y}{s} \right) \quad \dots (5.5)$$

where $\Re(c) < R(\ell) > 0, \Re(l) > 0, \Re(m) > 0$ and $\Re(\mathcal{P}) \geq 0$.

Proof. In order to prove (5.5), by using the definition of Laplace transform, the LHS of (5.5) becomes

$$\mathcal{L} \left\{ z^{l-1} \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; yz); s \right\} = \int_0^{\infty} z^{l-1} e^{-sz} \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; yz) dz$$

Now using the definition (1.12), we get

$$\mathcal{L} \left\{ z^{l-1} \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; yz); s \right\} = \int_0^{\infty} z^{l-1} e^{-sz} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell+n, c-\ell; \mathcal{P}) (yz)^n}{B(\ell, c-\ell)} \frac{1}{n!} dz$$

Interchanging the order of integration and summation and using Laplace transform, we obtain

$$\mathcal{L} \left\{ z^{l-1} \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; yz); s \right\} = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell+n, c-\ell; \mathcal{P}) y^n}{B(\ell, c-\ell)} \frac{1}{n!} \left\{ \int_0^{\infty} z^{l+n-1} e^{-sz} dz \right\}$$

$$= \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell+n, c-\ell; \mathcal{P}) y^n}{B(\ell, c-\ell)} \frac{1}{n!} \frac{\Gamma(l+n)}{s^{l+n}}$$

Now, making use of (4) and after straightforward calculation we finally arrive at (5.5).

Theorem 9. (Whittaker Transform)

$$\int_0^{\infty} t^{\rho-1} e^{-\delta t/2} \mathcal{W}_{\lambda, \mu}(\delta t) \left\{ \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; \omega t) \right\} dt$$

$$= \delta^{-\rho} \frac{\Gamma\left(\frac{1}{2} + \mu + \rho\right) \Gamma\left(\frac{1}{2} - \mu + \rho\right)}{\Gamma(1 - \lambda + \rho)} {}_2\mathcal{F}_{\mathcal{P}}^{(k_{\ell})} \left[a, \left(\frac{1}{2} + \mu + \rho\right), \Gamma\left(\frac{1}{2} - \mu + \rho\right), \ell; \frac{\omega}{s} \right] \quad \dots (5.6)$$

where $\Re(\mu \pm \rho) > 1/2$ and $\mathcal{W}_{\lambda, \mu}$ is the Whittaker confluent hypergeometric function and $\Re(c) < R(\ell) > 0, \Re(\rho) > 0, \Re(\delta) > 0$ and $\Re(\mathcal{P}) \geq 0$.

Proof. To prove Theorem 9, we use definition (1.12) in LHS of (5.6), we get

$$\begin{aligned} & \int_0^{\infty} t^{\rho-1} e^{-\delta t/2} \mathcal{W}_{\lambda, \mu}(\delta t) \left\{ \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; \omega t) \right\} dt \\ &= \int_0^{\infty} t^{\rho-1} e^{-\delta t/2} \mathcal{W}_{\lambda, \mu}(\delta t) \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P})}{B(\ell, c - \ell)} \frac{(\omega t)^n}{n!} dt \end{aligned}$$

Setting $\vartheta = \delta t$, we get

$$\begin{aligned} & \int_0^{\infty} t^{\rho-1} e^{-\delta t/2} \mathcal{W}_{\lambda, \mu}(\delta t) \left\{ \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; \omega t) \right\} dt \\ &= \int_0^{\infty} \left(\frac{\vartheta}{\delta}\right)^{\rho-1} e^{-\vartheta/2} \mathcal{W}_{\lambda, \mu}(\vartheta) \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P})}{B(\ell, c - \ell)} \frac{\omega^n \vartheta^n}{\delta^n (n!)} d\vartheta \end{aligned}$$

Interchanging the order of integration and summation, we get

$$\begin{aligned} & \int_0^{\infty} t^{\rho-1} e^{-\delta t/2} \mathcal{W}_{\lambda, \mu}(\delta t) \left\{ \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; \omega t) \right\} dt \\ &= \delta^{-\rho} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P})}{B(\ell, c - \ell)} \frac{\omega^n}{\delta^n (n!)} \left\{ \int_0^{\infty} \vartheta^{\rho+n-1} e^{-\vartheta/2} \mathcal{W}_{\lambda, \mu}(\vartheta) d\vartheta \right\} \end{aligned}$$

Next, using the Whittaker transform (5.3), we get

$$\begin{aligned} & \int_0^{\infty} t^{\rho-1} e^{-\delta t/2} \mathcal{W}_{\lambda, \mu}(\delta t) \left\{ \mathcal{F}_{\mathcal{P}}^{(k_{\ell})}(a, \ell; c; \omega t) \right\} dt \\ &= \delta^{-\rho} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\ell})}(\ell + n, c - \ell; \mathcal{P})}{B(\ell, c - \ell)} \frac{\omega^n}{\delta^n (n!)} \frac{\Gamma\left(\frac{1}{2} + \mu + \rho + n\right) \Gamma\left(\frac{1}{2} - \mu + \rho + n\right)}{\Gamma(1 - \lambda + \rho + n)} \end{aligned}$$

Interpreting the right-hand side of the above equation, in the view of the definition (1.12), we arrive at the required result (5.6).

Pathway Fractional Integration of the Extended Generalized Gauss Hypergeometric Function

The pathway fractional integral operator was introduced and studied by Mathai [13] and Nair and developed further by Mathai and Haubold ([27, 28]) as follows.

Let $f \in L(a, b)$, $\eta \in \mathbb{C}$, $d \in \mathbb{R}^+$ with $\Re(\eta) > 0$, $a > 0$, and $\alpha < 1$ be the pathway parameter. Then

$$\left(P_{0^+}^{(\eta, \alpha, d)} f \right)(t) = t^{\eta} \int_0^{\left[\frac{t}{d(1-\alpha)}\right]} \left[1 - \frac{d(1-\alpha)\tau}{t} \right]^{\frac{\eta}{1-\alpha}} f(\tau) d\tau. \quad \dots (6.1)$$

where $L(a, b)$ is the set of Lebesgue measurable function defined on (a, b) . For a real scalar α , the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(t) = c|t|^{\gamma-1} [1 - d(1-\alpha)|t|^\delta]^{\frac{\lambda}{1-\alpha}} \quad \dots (6.2)$$

provided that $-\infty < t < \infty, \delta > 0, \lambda \geq 0, [1 - d(1-\alpha)|t|^\delta] > 0$ and $\gamma > 0$. Here c is the normalizing constant and α is called the pathway parameter.

For $\alpha > 0$, (6.1) can be written as follows:

$$\left(P_{0^+}^{(\eta, \alpha, d)} f \right) (t) = t^\eta \int_0^{\left[\frac{t}{-d(1-\alpha)} \right]} \left[1 + \frac{d(\alpha-1)\tau}{t} \right]^{-\frac{\eta}{1-\alpha}} f(\tau) d\tau. \quad \dots (6.3)$$

For more details on the pathway model and its particular cases, the interested reader may refer to the recent work ([14], [24], [27, 28]). It is observed that the pathway fractional integral operator (6.1) can lead to other interesting examples of fractional calculus operators regarding some probability density functions and applications in statistics.

Our main result in this section is based on the following assertion giving a composition formula of the pathway fractional integral operator (6.1) with a power function (see Nair [14, Lemma 9]).

Lemma 9. Let $\eta \in \mathbb{C}, \Re(\eta) > 0, \beta \in \mathbb{C}, d \in \mathbb{R}^+$ and $\alpha < 1$. If $\Re(\beta) > 0$ and $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$, then we have

$$\left(P_{0^+}^{(\eta, \alpha, d)} [t^{\beta-1}] \right) (x) = \frac{x^{\eta+\beta} \Gamma(\beta) \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[d(1-\alpha)]^\beta \Gamma\left(1 + \frac{\eta}{1-\alpha} + \beta\right)} \quad \dots (6.4)$$

Now we are ready to present our result which is composition formula of the pathway fractional integration operator (6.1) with a product of the extended generalized Gauss hypergeometric function (1.12) asserted by the following theorem.

Theorem 10. Let $\alpha < 1, \eta, \rho, e \in \mathbb{C}$ with $\Re(\eta) > 0, \Re(\rho) > 0$ and $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$. Also let $\Re(\mathcal{P}) \geq 0, \Re(c) < \Re(\ell) > 0$, then there holds the following formula:

$$\begin{aligned} & \left(P_{0^+}^{(\eta, \alpha, d)} \left[t^{\rho-1} {}_1\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(a, \ell; c; et) \right] \right) (x) \\ &= \frac{x^{\eta+\rho} \Gamma(\rho) \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[d(1-\alpha)]^\rho \Gamma\left(1 + \frac{\eta}{1-\alpha} + \rho\right)} \times {}_1\mathcal{F}_{\mathcal{P}+1}^{(k_\ell)} \left[\begin{matrix} a, \ell, \rho \\ c, \left(1 + \frac{\eta}{1-\alpha} + \rho\right) \end{matrix} ; \frac{ex}{d(1-\alpha)} \right] \quad \dots (6.5) \end{aligned}$$

Proof. By Applying (1.12), we have

$$\begin{aligned} & \left(P_{0^+}^{(\eta, \alpha, d)} \left[t^{\rho-1} {}_1\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(a, \ell; c; et) \right] \right) (x) = \left(P_{0^+}^{(\eta, \alpha, d)} \left[t^{\rho-1} \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_\ell)}(\ell+n, c-\ell; \mathcal{P})}{B(\ell, c-\ell)} \frac{e^n t^n}{n!} \right] \right) (x) \\ &= \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_\ell)}(\ell+n, c-\ell; \mathcal{P})}{B(\ell, c-\ell)} \frac{e^n}{n!} \left(P_{0^+}^{(\eta, \alpha, d)} [t^{\rho+n-1}] \right) (x) \end{aligned}$$

Now applying Lemma 9 with β replaced by $\rho + n$ to the pathway integral, we obtain the following result

$$\left(P_{0+}^{(\eta, \alpha, d)} \left[t^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})}(a, b; c; et) \right] \right) (x) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\mathcal{P}}^{(k_{\mathcal{P}})}(b+n, c-b; \mathcal{P})}{B(b, c-b)} \frac{e^n}{n!} \\ \times x^{\eta+\rho+n} \frac{\Gamma(\rho+n) \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[d(1-\alpha)]^{\rho+n} \Gamma\left(1 + \frac{\eta}{1-\alpha} + \rho+n\right)}.$$

then, in view of (1.12), is easy to arrive at the expression (6.5). This completes the proof.

If we set $\alpha = 0$ and $d = 1$ and replacing η by $\eta - 1$ in (6.5), then we have the following relationship:

$$\left(P_{0+}^{(\eta-1, 0, 1)} f \right) (t) = \int_0^t (t-\tau)^{\eta-1} f(\tau) d\tau = \Gamma(\eta) [I_{0+}^{\eta} f](t) \quad (\Re(\eta) > 0) \quad \dots (6.6)$$

We obtain fractional integral formula involving left-sided Riemann-Liouville fractional integral operator stated in the next corollary below.

Corollary 3. Let the parameters $\eta, \rho, e \in \mathbb{C}$ with $\Re(\eta) > 0, \Re(\rho) > 0, d > 0$. Also let $\Re(\mathcal{P}) \geq 0, \Re(c) < \Re(b) > 0$. Then we have the following relation:

$$\left(I_{0+}^{\eta} \left[x^{\rho-1} \mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})}(a, b; c; ex) \right] \right) = \frac{x^{\eta+\rho-1} \Gamma(\rho) \Gamma(\eta)}{\Gamma(\eta+\rho)} \times {}_1\mathcal{F}_{\mathcal{P}+1}^{(k_{\mathcal{P}})} \left[\begin{matrix} a, b, \rho \\ c, \Gamma(\eta+\rho) \end{matrix}; ex \right] \dots (6.7)$$

Concluding Remarks and Observation

It is noted that the results obtained here are useful in deriving various fractional integral operators for each of the families of the extended generalized hypergeometric functions $\mathcal{F}_{\mathcal{P}}^{(k_{\mathcal{P}})}(\cdot)$ defined in (1.12). It can be easily see that if we set $\alpha = 0, d = 0$, and $f(t)$ is replaced by

$${}_2F_1 \left(\eta + \beta, -\gamma; \eta; 1 - \frac{t}{x} \right) f(t)$$

(1.12) yields the Saigo fractional integral operator. Thus we can obtained the generalization of left-sided fractional integrals, like Saigo, Erdélyi-Kober (see [1]; see also [30]), and so on, by suitable substitutions. Therefore, the results presented here are easily shown to be converted to those corresponding to the above well known fractional operators. Laplace, Beta and Whittaker transforms for extended hypergeometric functions are obtained as common converge. Moreover, we investigated a composition of the pathway fractional integral operator with a product extended generalized hypergeometric function (1.12).

It can be seen that the results obtained in this paper are new and effective mathematical tool and, also extension of many results in literature.

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