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Fractional Calculus Results Based On Extended Gauss Hypergeometric Functions and Via Pathway Model

Sunil Kumar Sharma#1, Deepti Arora#2, Kanishk Sharma#3
Department of Mathematics
Arya College of Engineering and Research Centre, Jaipur, Rajasthan, India
Email: sunil94147@gmail.com
Department of Mathematics
Jaipur Engineering College, Jaipur, Rajasthan, India
Email: aroradeepti591@gmail.com

Abstract

In this paper, we are discussing the formula of the so-called fractional integration of the Nair-based pathway with a limited product of altimeter functions. We have also obtained two complementary fractional formulas that include the integrated operators Riemann-Liouville on the left side. All the results obtained here are of a general nature and can produce many results in the theory of special functions. In addition, using some integrated conversion in the resulting formulas, we offer some other image formats. All derived results here are of a general nature and can produce a series of results (known and new) in the theory of special functions.

Keywords: Pathway fractional integral operator, Extended hypergeometirc functions, Hypergeometric function, Integral Transforms.

Introduction

Operators, which included several special functions, found importance and important applications in the various subfields of the applicable mathematical analysis. In recent years, small calculations have become one of the most advanced parts of science and mathematics. From the last four decades, a number of researchers such as Choi et al. [4], Saigo [1, 2], Khan et al. [3], Kiryakova [5,6], Kilbas [7, 8], Machado et al. [9, 10], Kalla [11, 12], Kalla and Saxena [13] studied the characteristics, the applications and the various extensions of different operators for hypergeometric. Various applications of operators can be found, for example, in turbulence and fluid dynamics, fractal engine models, random dynamic order, plasma physics, controlled nuclear thermal integration, non-linear biological systems and astrophysics.

During this search $\mathbb{N}, \Re, \mathbb{C}$ and $\mathbb{Z}_0^-$ refer to sets of positive integers, real numbers, compound numbers and non-positive integers respectively and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Chaudhary et al. [14] presented the following classical Beta function $B(\alpha, \beta)$ as :

$$B(\alpha, \beta; \mathcal{P}) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \exp\left(-\frac{\mathcal{P}}{t(1-t)}\right) dt \qquad \dots(1)$$

$$(\min\{\Re(\alpha), \Re(\beta)\} > 0;\ \Re(\mathcal{P}) \geq 0)$$

Here Beta function is a function of two variables α and β defined by

$$B(\alpha,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0,\Re(\beta)>0)\\[2ex] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\backslash\mathbb{Z}_0^-)\end{cases}\qquad \dots(2)$$

It can be seen that Γ is the acquainted Gamma function.

At the latest, Srivastava et al. [16] shown a natural generalization in terms of the function $\Xi\left(\{k_\ell\}_{\ell\in N_0};z\right)$ defined as follows (see [16]):

$$B_{\mathcal{P}}^{(\{k_\ell\}_{\ell\in N_0})}(x,y;\mathcal{P})=\int_0^1 t^{x-1}(1-t)^{y-1}\,\Xi\left(\{k_\ell\}_{\ell\in N_0};-\frac{\mathcal{P}}{t(1-t)}\right)dt \qquad \dots(3)$$

$$(\Re(\mathcal{P})\geq 0;\min\{\Re(x),\Re(y)\}>0).$$

and

$$\mathcal{F}_{\mathcal{P}}^{(\{k_\ell\}_{\ell\in N_0})}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{\mathcal{P}}^{(\{k_\ell\}_{\ell\in N_0})}(b+n,c-b;\mathcal{P})}{B(b,c-b)}\frac{z^n}{n!} \qquad \dots(4)$$

$$(|z|<1;\ \Re(c)>R(b)>0;\Re(\mathcal{P})\geq 0)$$

where $\Xi\left(\{k_\ell\}_{\ell\in N_0};z\right)$ is given in (see [16]):

The detailed and methodological study described above was determined primarily by the apparent probability of applications in the most common Gothic functions and their particular states in many areas of mathematical, physical, technical and statistical sciences (see [16]).

Let $f(x)=L(a,b),\alpha\in\mathbb{C},\Re(\alpha)>0$, Riemann-Liouville fractional integral operator of left sided is follwing

$$\left(I_{0_+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt \qquad \dots(5)$$

For more details, we refer to Kiryakova [5], Samko-Kilbas-Marichev [8] etc.

If f (t) is replaced by t ^ γ f (t) in (5), then the operator Erde lyi-Kobeb, if it replaces by ${}_2F_1\left(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}\right)f(t)$, then (1) takes the form of Saigo hypergeometric fractional integral

$$\frac{\Gamma(\eta)}{x^{-\eta-\beta}}I_{0_+}^{\eta,\beta,\gamma}f(x)=\int_0^x (x-t)^{\eta-1}\ {}_2F_1\left(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}\right)f(t)\,dt$$

Many other operators generalized fractional calculus can be obtained if on the place of $f(t)$ one takes $\phi(t)$ as it is done in Kiryakova [11] for a Fox's H-function $\phi(t)=H_{m,m}^{m,0}(t)$.

Pathway Fractional Integration of the Extended Generalized Gauss Hypergeometric Function

The pathway operator is established by[19, p.239] as:

$$\left(P_{0+}^{(\eta,\lambda,d)}f\right)(x)=x^{\eta}\int_{0}^{\left[\frac{x}{d(1-\lambda)}\right]}\left[1-\frac{d(1-\lambda)t}{x}\right]^{\frac{\eta}{1-\lambda}}f(t)dt, \quad \dots(6)$$

where $f(x)\in L(a,b), \eta\in\mathbb{C}, \ \Re(\eta)>0$, d>0 and 'pathway parameters' $\lambda<1$.

In this paper, we will discuss only the condition of pathway parameters $\lambda<1$. Taking as $\lambda\longrightarrow 1$ (6) receive the following exponential form,

$$\lim_{\lambda\to 1} c|x|^{\gamma-1}\left[1-d(1-\lambda)|x|^{\delta}\right]^{\frac{\eta}{1-\lambda}}=\lim_{x\to 1} c|x|^{\gamma-1}\left[1-d\,\eta\,|x|^{\delta}\,\frac{(1-\lambda)}{\eta}\right]^{\frac{\eta}{1-\lambda}}$$

$$=c|x|^{\gamma-1}\exp\left(-d\,\eta\,|x|^{\delta}\right). \quad \dots(7)$$

It includes Gamma, Weibull, Chi-Square, Laplace, Maxwell-Boltzmann and several related densities. Therefore, the operator introduced in this article may be associated with different types of statistical densities.

When $\lambda\longrightarrow 1_{-}$, $\left[1-\frac{d(1-\lambda)t}{x}\right]^{\frac{\eta}{1-\lambda}}\longrightarrow e^{-d\eta t}$. Then, the operators (6) converts in the Laplace integral transform of f with the parameters $\frac{d\eta}{x}$:

$$\left(P_{0+}^{(\eta,1)}f\right)(x)=x^{\eta}\int_{0}^{\infty}e^{-\frac{d\eta}{x}}f(t)dt=x^{\eta}\,L_f\left(\frac{d\eta}{x}\right). \quad \dots(8)$$

The integral fractional path operator introduced by Matai [13] is investigated as follows. Let $f\in L(a,b)$, $\eta\in\mathbb{C}, d\in\Re^{+}$ with $\Re(\eta)>0$, $a>0$, and $\alpha<1$ be the pathway parameter. It should be noted that the exponential path operator (6) may lead to other interesting examples of sub-calculus operators in terms of some probability density functions and their applications in statistics.

Our main result in this section is based on the following statement, which introduces the combined formula of the integral fractional path operator (6) with a power function (see Nier [19, cf. 9]).

Lemma 1. Let $\eta\in\mathbb{C}, \Re(\eta)>0, \beta\in\mathbb{C}, d\in\Re^{+}$ and $\alpha<1$. If $\Re(\beta)>0$ and $\Re\left(\frac{\eta}{1-\alpha}\right)>-1$, then we have

$$\left(P_{0^{+}}^{(\eta,\alpha,d)}\left[t^{\beta-1}\right]\right)(x)=\frac{x^{\eta+\beta}\,\Gamma(\beta)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[d(1-\alpha)]^{\beta}\,\Gamma\left(1+\frac{\eta}{1-\alpha}+\beta\right)} \quad \dots(9)$$

Now, we are ready to present our results, which represent the configuration formula of the pathway integration operator (6) with a product of the extended generalized Gaussian hypergeometric function (3), which is confirmed by the following theory.

Theorem 1. Let $\alpha < 1, \eta, \rho, e \in \mathbb{C}$ with $\Re(\eta) > 0, \Re(\rho) > 0$ and $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$. Also let $\Re(\mathcal{P}) \geq 0, \Re(c) < R(\mathscr{b}) > 0$, then we find interesting the following formula:

$$\left(P_{0^+}^{(\eta,\alpha,d)}\left[t^{\rho-1}\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(a,\mathscr{b};c;et)\right]\right)(x)$$

$$= \frac{x^{\eta+\rho}\,\Gamma(\rho)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[d(1-\alpha)]^{\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho\right)} \times {}_1\mathcal{F}_{\mathcal{P}+1}^{(k_\ell)}\left[\begin{matrix} a,\mathscr{b},\rho \\ c,\left(1+\frac{\eta}{1-\alpha}+\rho\right); \end{matrix} ;\ \frac{ex}{d(1-\alpha)}\right] \quad \ldots(10)$$

Proof. By Applying (6), we have

$$\left(P_{0^+}^{(\eta,\alpha,d)}\left[t^{\rho-1}\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(a,\mathscr{b};c;et)\right]\right)(x) = \left(P_{0^+}^{(\eta,\alpha,d)}\left[t^{\rho-1}\sum_{n=0}^{\infty}(a)_n \frac{B_{\mathcal{P}}^{(k_\ell)}(\mathscr{b}+n,c-\mathscr{b};\mathcal{P})}{B(\mathscr{b},c-\mathscr{b})}\frac{e^n t^n}{n!}\right]\right)(x)$$

$$= \sum_{n=0}^{\infty}(a)_n \frac{B_{\mathcal{P}}^{(k_\ell)}(\mathscr{b}+n,c-\mathscr{b};\mathcal{P})}{B(\mathscr{b},c-\mathscr{b})}\frac{e^n}{n!}\left(P_{0^+}^{(\eta,\alpha,d)}\left[t^{\rho+n-1}\right]\right)(x)$$

Now applying Lemma 1 with β replaced by $\rho + n$ to above integral, we obtain the following result

$$\left(P_{0^+}^{(\eta,\alpha,d)}\left[t^{\rho-1}\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(a,\mathscr{b};c;et)\right]\right)(x) = \sum_{n=0}^{\infty}(a)_n \frac{B_{\mathcal{P}}^{(k_\ell)}(\mathscr{b}+n,c-\mathscr{b};\mathcal{P})}{B(\mathscr{b},c-\mathscr{b})}\frac{e^n}{n!}$$

$$\times x^{\eta+\rho+n}\frac{\Gamma(\rho+n)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[d(1-\alpha)]^{\rho+n}\,\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho+n\right)}.$$

next, in (6), it is simple to reach at the expression (10). This completes the proof.

If we apply the parameters $\alpha = 0$ and $d = 1$ and changing η by $\eta - 1$ in (10), then we make the another relationship:

$$\left(P_{0^+}^{(\eta-1,0,1)}f\right)(t) = \int_0^t (t-\tau)^{\eta-1}f(\tau)d\tau = \Gamma(\eta)\left[\left(I_{0^+}^{\eta}f\right)(t)\right] \qquad (\Re(\eta) > 0) \qquad \ldots(11)$$

We find a integral formula that includes the fractional integration factor on the right-hand side of Riemann-Liouville mentioned in the following result.

Corollary 1. Assuming parameters as $\eta, \rho, e \in \mathbb{C}$ with $\Re(\eta) > 0, , \Re(\rho) > 0, d > 0$. Also let $\Re(\mathcal{P}) \geq 0, \Re(c) < R(\mathscr{b}) > 0$.Then we have the following relation:

$$\left(I_{0^+}^{\eta}\left[\ x^{\rho-1}\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(a,\mathscr{b};c;ex)\right]\right) = \frac{x^{\eta+\rho-1}\,\Gamma(\rho)\,\Gamma(\eta)}{\Gamma(\eta+\rho)} \times {}_1\mathcal{F}_{\mathcal{P}+1}^{(k_\ell)}\left[\begin{matrix} a,\mathscr{b},\rho \\ c,\Gamma(\eta+\rho); \end{matrix} ;\ ex\right] \ldots(12)$$

Integral Transform of the Extended Generalized Hypergeometric Functions

Here, the image of $\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(.)$ under Beta, Laplace and Whittaker functions have been obtained. For our purpose, we begin by recalling some integral transforms.

The Euler (Beta) transform of $f(z)$ is defined as [20]:

$$B\{f(z):\alpha,\beta\} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\,dz \qquad (\Re(\alpha)>0, \Re(\beta)>0) \qquad \ldots(13)$$

Laplace transform of $f(z)$ is defined as [20]:

$$\mathcal{L}\{f(z):s\} = \int_0^\infty e^{-sz} f(z)\,dz \qquad (\Re(s)>0\,) \qquad \ldots(14)$$

The Whittaker transform is defined as [20]:

$$\int_0^\infty e^{-t/2} t^{\vartheta-1} \mathcal{W}_{\xi,\mu}(t)dt = \frac{\Gamma\left(\frac{1}{2}+\mu+\vartheta\right)\,\Gamma\left(\frac{1}{2}-\mu+\vartheta\right)}{\Gamma(1-\xi+\vartheta)} \qquad \ldots(15)$$

where $\Re(\mu\pm\vartheta) > {}^{-1}/_{2}$ and $\mathcal{W}_{\xi,\mu}(t)$ is the Whittaker confluent hypergeometric function.

Theorem 2. (Beta function)

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_\xi)}(l+m,\mathscr{b};c;yz) : l,m\right\} = B(l,m).\mathcal{F}_{\mathcal{P}}^{(k_\xi)}(l,\mathscr{b};c;y) \qquad \ldots(16)$$

where $\Re(c) < R(\mathscr{b}) > 0, \Re(l)>0, \Re(m)>0$ and $\Re(\mathcal{P}) \geq 0.$

Proof. In order to prove (16), by applying Beta transform (13), the LHS of (16) converts

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_\xi)}(l+m,\mathscr{b};c;yz) : l,m\right\} = \int_0^1 z^{l-1}(1-z)^{m-1}\left\{\mathcal{F}_{\mathcal{P}}^{(k_\xi)}(l+m,\mathscr{b};c;yz)\right\}dz$$

Now using the definition (13), we get

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_\xi)}(l+m,\mathscr{b};c;yz) : l,m\right\} = \int_0^1 z^{l-1}(1-z)^{m-1}\sum_{n=0}^\infty (l+m)_n \frac{B_{\mathcal{P}}^{(k_\xi)}(\mathscr{b}+n,c-\mathscr{b};\mathcal{P})}{B(\mathscr{b},c-\mathscr{b})}\frac{(yz)^n}{n!}dz$$

Reciprocate the sequence of integration and summation and making use of beta integral, then

$$B\left\{\mathcal{F}_{\mathcal{P}}^{(k_\xi)}(l+m,\mathscr{b};c;yz): l,m\right\}$$

$$= \sum_{n=0}^\infty (l+m)_n \frac{B_{\mathcal{P}}^{(k_\xi)}(\mathscr{b}+n,c-\mathscr{b};\mathcal{P})}{B(\mathscr{b},c-\mathscr{b})}\frac{y^n}{n!}\left\{\int_0^1 z^{l+n-1}(1-z)^{m-1}\right\}dz$$

$$= \sum_{n=0}^\infty (l+m)_n \frac{B_{\mathcal{P}}^{(k_\xi)}(\mathscr{b}+n,c-\mathscr{b};\mathcal{P})}{B(\mathscr{b},c-\mathscr{b})}\frac{y^n}{n!}\,\frac{\Gamma(l+n)\Gamma(m)}{\Gamma(l+n+m)}$$

Then, using (9) and the correct interpretation of the equation above, in terms of the definition (6), we arrive at the result (16).

Theorem 3. (Laplace Transform)

$$\mathcal{L}\left\{z^{l-1}\,\mathcal{F}_{\mathcal{P}}^{(k_\xi)}(a,\mathscr{b};c;yz): s\right\} = \frac{\Gamma(l)}{s^l}\,{}_1\mathcal{F}_{\mathcal{P}}^{(k_\xi)}\left(a,l,\mathscr{b};c;\frac{y}{s}\right) \qquad \ldots(17)$$

where $\Re(c) < R(\mathscr{b}) > 0, \Re(l) > 0, \Re(m) > 0$ and $\Re(\mathcal{P}) \geq 0$.

Theorem 4. (Whittaker Transform)

$$\int_0^\infty t^{\rho-1} e^{-\delta t/2}\, \mathcal{W}_{\lambda,\mu}(\delta t)\left\{\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(a, \mathscr{b}; c; \omega t)\right\} dt$$

$$= \delta^{-\rho} \frac{\Gamma\left(\frac{1}{2}+\mu+\rho\right)\Gamma\left(\frac{1}{2}-\mu+\rho\right)}{\Gamma(1-\lambda+\rho)}\ {}_2\mathcal{F}_{\mathcal{P}}^{(k_\ell)}\left[\begin{matrix} a, \left(\frac{1}{2}+\mu+\rho\right), \Gamma\left(\frac{1}{2}-\mu+\rho\right), \mathscr{b}; \\ c, (1-\lambda+\rho); \end{matrix}\ \frac{\omega}{s}\right] \quad \ldots (18)$$

where $\Re(\mu \pm \rho) > {}^1/_2$ and $\mathcal{W}_{\lambda,\mu}$ is the Whittaker confluent hypergeometric function and $\Re(c) < R(\mathscr{b}) > 0, \Re(\rho) > 0, \Re(\delta) > 0$ and $\Re(\mathcal{P}) \geq 0$.

Concluding Remarks and Observation

We conclude this study by pointing out that the results obtained here are of a general nature and make it possible to extract multiple integrated formulas in the integrated pathway fractional integral. We have studied the fractional composition of the pathway fractional integral operator with a product extended generalized hypergeometric function.

The results obtained here are useful for deriving many complementary integrator operators for each family of extended extended hypergeometric functions $\mathcal{F}_{\mathcal{P}}^{(k_\ell)}(.)$

It can be easily observe that if we take parameters as $\alpha = 0, d = 0$, and $f(t)$ is changing by

$${}_2F_1\left(\eta+\beta, -\gamma; \eta; 1-\frac{t}{x}\right) f(t)$$

got interesting the Saigo operator for fractional calculus. Laplace, Beta and Whittaker transforms for extended hypergeometric functions are obtained as common converge. Moreover, Results derived in this paper are very significant and may find application in the solution of fractional order differential equations that are arising in theory of special functions.

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