

Basic Analogue of Fractional Derivative and Some special Functions

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Abstract

This paper is devoted to the basic analogues of fractional differentiation of classical functions and a theorem on term by term basic analogues of fractional differentiation has been derived.

Keywords: Analogue, Fractional, Calculus, Integrals, Arbitrary.

1. Introduction

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades Fractional Calculus has been applied to almost every field of Mathematics like Special Functions etc., Science, Engineering and Technology. Many applications of Fractional Calculus can be found in Turbulence and Fluid Dynamics, Stochastic dynamical System, Plasma Physics and Controlled Thermonuclear Fusion, Non-linear Control Theory,

2. Definition1. The fractional q -differential operator of order α is defined as follows

$$D_{x,q}^{\alpha} f(x) = \frac{1}{\Gamma_q(-\alpha)} \int_0^x (x-yq)_{-\alpha-1} f(y) d(y;q) \quad \dots\dots\dots (1.1)$$

Where $\text{Re}(\alpha) < 0$.

As a particular case of (1.1), we have

$$D_{x,q}^{\alpha} x^{\mu-1} = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1} \quad \dots\dots\dots (1.2)$$

The Wright generalized hyper geometric function is given by Kilbas (2005) [4] ${}_p\Psi_q(z)$

$${}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} ; z \right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + rA_j) z^r}{\prod_{j=1}^q \Gamma(b_j + rB_j) r!} \quad \dots\dots\dots (1.3)$$

$$A_j > 0 (j = 1, 2, \dots, p), B_j > 0 (j = 1, 2, \dots, q); 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geq 0 \text{ (Equality only for approximately bounded } z \text{)}].$$

3. Main Results

In this section we shall prove a theorem on term by term fractional q -differentiation of a power series.

Theorem 1: If the series $\sum_{n=0}^{\infty} \frac{a_n x^n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})}$ converges absolutely for $|x| < \rho$ then

$$D_{x,q}^{\mu} \left\{ x^{\lambda-1} \sum_{n=0}^{\infty} \frac{a_n x^n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})!} \right\} = \sum_{n=0}^{\infty} \frac{a_n D_{x,q}^{\mu} \{x^{\lambda+n-1}\}}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})} \dots\dots\dots (1.4)$$

provided $\text{Re}(\lambda) > 0, \text{Re}(\mu) < 0, 0 < |q| < 1$.

Proof: In view of (1.3), we have

$$\begin{aligned} &= \frac{1}{\Gamma_q(-\mu)} \int_0^x (x-yq)_{-\mu-1} y^{\lambda-1} \sum_{n=0}^{\infty} \frac{a_n y^n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})} d(y;q) \\ &= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \int_0^1 t^{\lambda-1} (1-tq)_{-\mu-1} \sum_{n=0}^{\infty} \frac{a_n t^n x^n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})!} d(t;q) \dots\dots\dots (1.5) \end{aligned}$$

Now the following observations are made

- (i) $\sum_{n=0}^{\infty} \frac{a_n t^n x^n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})}$ Converges absolutely and therefore uniformly in domain of x over the region of integration.
- (ii) $\int_0^1 |t^{\lambda-1} (1-tq)_{-\mu-1}| d(t;q)$ is convergent, provided $\text{Re}(\lambda) > 0, \text{Re}(\mu) < 0, 0 < |q| < 1$.

Therefore the order of integration and summation can be interchanged in (1.3) to obtain

$$= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{n=0}^{\infty} \frac{a_n x^n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})} \int_0^1 t^{\lambda+n-1} (1-tq)_{-\mu-1} d(t; q)$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{\prod_{j=1}^m \Gamma(\mu_j + \frac{n}{\rho_j})} D_{x,q}^{\mu} \{x^{\lambda+n-1}\} \dots\dots\dots (1.6)$$

This proves the theorem (1.4)

4. Special Cases

Let us consider some special cases of above result:

(i) If we put $m=1, \rho_j = \frac{1}{\alpha}, \mu_j = \beta$ in (1.4), we get

$$D_{x,q}^{\mu} \left\{ x^{\lambda-1} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \right\} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} D_{x,q}^{\mu} \{x^{\lambda+n-1}\} \dots\dots\dots (1.7)$$

Which is the fractional q- derivative of generalized Mittag-Leffler function denoted by $E_{\alpha, \beta}(z)$.

(ii) If we take $m=2, a_n=1$ in (1.4) it reduces to

$$D_{x,q}^{\mu} \left\{ x^{\lambda-1} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu_1 + \frac{n}{\rho_1}) \Gamma(\mu_2 + \frac{n}{\rho_2})} \right\} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu_1 + \frac{n}{\rho_1}) \Gamma(\mu_2 + \frac{n}{\rho_2})} D_{x,q}^{\mu} \{x^{\lambda+n-1}\} \dots\dots\dots (1.8)$$

Which is the fractional q-derivative of the function shown by Dzrbashjan [2] that it is an entire function of order?

$$\rho = \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} \text{ and type } \sigma = \left(\frac{\rho_1}{\rho} \right)^{\frac{\rho}{\rho_1}} \left(\frac{\rho_2}{\rho} \right)^{\frac{\rho}{\rho_2}}$$

(iii) If we take $m=1, \rho_j = \frac{1}{\alpha}, \mu_j = 1$ in (1.4), we get

$$D_{x,q}^{\mu} \left\{ x^{\lambda-1} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \right\} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)} D_{x,q}^{\mu} \{x^{\lambda+n-1}\} \dots\dots\dots (1.9)$$

Which is the fractional q- derivative of Mittag-Leffler function denoted by $E_{\alpha}(z)$.

5. References

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